Scienta Press Journals

American Scholar
Journal of Interdisciplinary Research and Knowledge

The Role Of Non-Standard Geometric Problems In Developing The Productive Thinking Of Middle And High School Students

A.X. Rakhmatullayev
Associate Professor, Nizami Tashkent State Pedagogical University
U.B. Matnazarov
Associate Professor, Nizami Tashkent State Pedagogical University
E-mail: batiyxannasirov@gmail.com

Abstract

This article examines the role of drawings in solving geometric problems as an effective tool for developing productive thinking among students. The use of visual representations contributes to a clearer understanding of problem conditions and facilitates the selection of appropriate solution methods. The study analyzes the significance of non-standard geometric problems in the formation of students' thinking skills. Drawings enable learners to visualize the given situation accurately, making it easier to identify solution strategies. Evaluating a visual representation of a problem is generally more effective than relying solely on textual information. Although drawings do not provide formal proofs of theorems, they play an important role in helping students understand mathematical concepts more easily and retain them in long-term memory.

Keywords: productive thinking, teaching methods, theorem, non-standard problems, proof, calculation, construction, proof by contradiction, geometric problems, drawing

Introduction

Geometric problems play a significant role in the development of students' productive thinking, particularly in upper secondary education, where learners are expected to prove fundamental theorems in solid geometry. However, students often experience difficulties in independently formulating theoretical conclusions derived from general mathematical reasoning and in solving proof-based and theoretical problems.

To enhance students' spatial imagination, it is advisable to provide exercises related to skew lines and the theorem of three perpendiculars through visual and illustrative representations. Before proving theorems, students must develop a solid conceptual understanding of geometry. If a teacher merely demonstrates a theorem and then assigns a similar theorem for practice, students are more likely to engage in reproductive rather than productive thinking. Productive thinking develops when learners independently generate new ideas and construct solutions based on their existing knowledge and experiences.

Literature Review

The issue of developing students' productive thinking has been extensively studied in psychology, pedagogy, and mathematics education. Modern educational paradigms emphasize not only the acquisition of mathematical knowledge but also the development of students' abilities to think independently, solve problems creatively, and justify their conclusions logically.

The theoretical foundations of productive thinking were established by Gestalt psychologists, particularly Max Wertheimer, who introduced the concept of productive thinking as a process of discovering new relationships and solving problems through insight rather than mechanical memorization. According to Wertheimer, meaningful learning occurs when students understand the structural relationships between concepts and independently construct solution strategies.

A significant contribution to the study of cognitive development was made by Lev Vygotsky. His sociocultural theory emphasizes that intellectual development occurs through active engagement in problem-solving activities and social interaction. Vygotsky argued that appropriately designed educational tasks can stimulate higher-order thinking processes and contribute to the development of learners' cognitive abilities.

The psychological aspects of mathematical thinking were further investigated by Jean Piaget, who demonstrated that the development of logical and spatial reasoning is closely associated with students' cognitive maturation. His studies showed that geometry provides a particularly effective environment for the formation of abstract reasoning and analytical thinking skills.

The role of problem-solving in mathematics education was comprehensively analyzed by George Pólya. In his influential work *How to Solve It*, Pólya proposed a system of heuristic methods that encourage students to analyze problem conditions, formulate hypotheses, develop solution plans, and verify results. His approach remains one of the most widely accepted theoretical foundations for developing productive mathematical thinking.

Research on mathematical abilities conducted by Vladimir A. Krutetskii demonstrated that successful mathematical performance depends on the development of logical reasoning, abstraction, generalization, and proof-construction skills. Krutetskii emphasized that non-routine and proof-oriented mathematical tasks play a crucial role in cultivating these abilities.

In geometry education, numerous studies have highlighted the importance of visual representations and diagrammatic reasoning. Geometric drawings help students perceive relationships between mathematical objects, improve spatial imagination, and facilitate the understanding of abstract concepts. Researchers have shown that visual models serve as powerful cognitive tools that support both conceptual understanding and problem-solving processes.

Contemporary mathematics education researchers increasingly emphasize the educational value of non-standard problems. Unlike routine exercises that require the application of familiar algorithms, non-standard problems encourage students to explore multiple solution pathways, employ creative reasoning, and apply knowledge in unfamiliar contexts. Such tasks contribute significantly to the development of critical thinking, flexibility, originality, and independence.

Several studies have also demonstrated that theorem proving is one of the most effective means of developing productive thinking in geometry. The process of proving requires students to analyze given information, establish logical relationships, formulate arguments, and justify

ISSN: 3071-1835 Volume 01, Issue 05, June, 2026

conclusions. Through these activities, learners develop deductive reasoning skills and a deeper understanding of mathematical structures.

Despite considerable research devoted to productive thinking, mathematical reasoning, and problem-solving, the specific role of non-standard geometric problems in developing productive thinking among middle and high school students remains insufficiently investigated. Existing studies primarily focus on general mathematical problem-solving or theorem proving, whereas the cognitive potential of non-standard geometric tasks, particularly those involving visual reasoning and spatial imagination, has received comparatively less attention.

Therefore, the present study aims to investigate the pedagogical significance of non-standard geometric problems and their contribution to the development of productive thinking among middle and high school students. Special attention is given to the role of geometric constructions, diagrams, theorem proving, and heuristic problem-solving strategies in fostering students' intellectual activity, creativity, and independent mathematical reasoning.

Main Part

For productive thinking to develop effectively, students must fully understand a theorem and be able to mentally visualize the corresponding geometric figure. Productive thinking is not limited to theorem proving and problem solving; it can also be fostered through the derivation of mathematical formulas and the investigation of problem situations.

Consider the following example.

Problem 1

Let the lines CD and EF intersect at point B in a plane α, where $CB = BD$ and $EB = BF$. Let point A be located outside plane α such that $AE = AF$ and $AC = AD$. Prove that line AB is perpendicular to plane α.

Proof. This problem can be solved using the method of analysis.

1. To prove that AB is perpendicular to plane α, it is sufficient to prove that AB is perpendicular to both CD and EF.
2. To prove that $AB \perp CD$, it is enough to show that triangles ABC and ABD are congruent.
3. Since $BC = BD$, $AC = AD$, and AB is common, triangles ABC and ABD are congruent by the SSS criterion. Therefore, the corresponding angles are equal, implying $AB \perp CD$.

American Scholar: Journal of Interdisciplinary Research and Knowledge
ISSN: 3071-1835 Volume 01, Issue 05, June, 2026

4. Similarly, to prove AB ⊥ EF, it is sufficient to show that triangles ABE and ABF are congruent.

5. Since BE = BF, AE = AF, and AB is common, triangles ABE and ABF are congruent. Consequently, AB is perpendicular to EF.

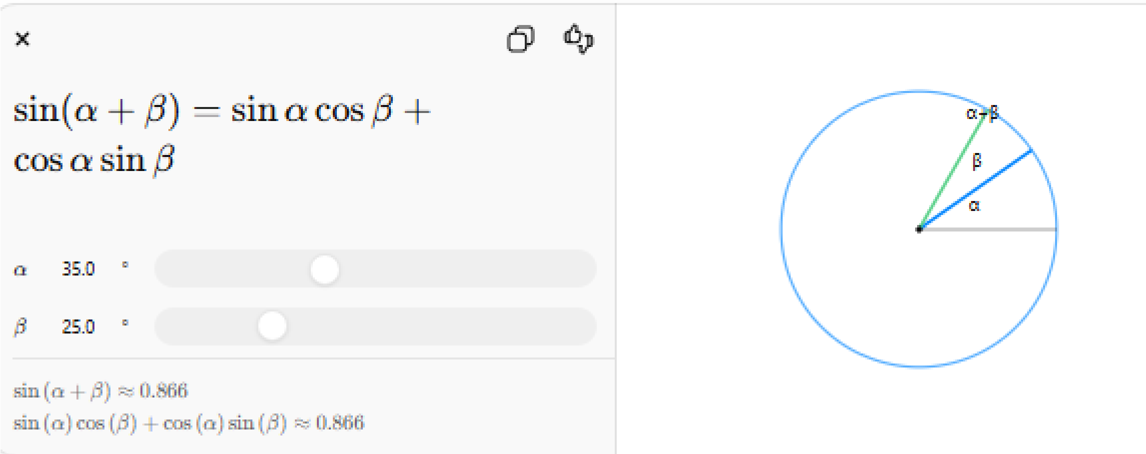
Since AB is perpendicular to two intersecting lines lying in plane α, it follows that AB ⊥ α.

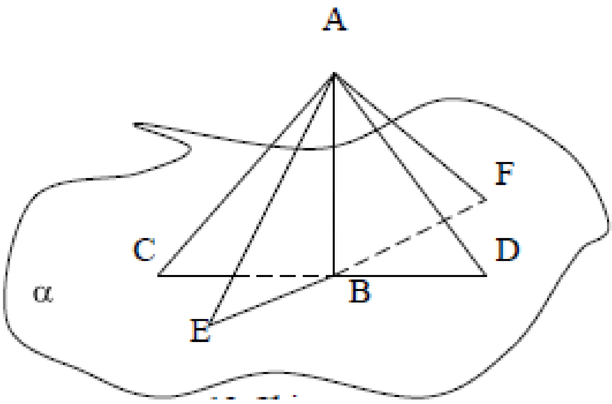
Derivation of the Sine Addition Formula

Suppose students are introduced to the topic *“The Sine of the Sum of Two Angles.”* After understanding the derivation, they can successfully apply the resulting formula to solve related problems.

Given a circle with center O and radius OA = OB = 1, let ∠EOB = α, ∠BOA = β, and ∠DOA = α + β.

The goal is to prove:





Using geometric relationships and properties of right triangles, it can be shown that:

- CD = EB,
- EB = sin α · cos β,
- AC = cos α · sin β,

and therefore:

sin(α + β) = AD = CD + AC

ISSN: 3071-1835 Volume 01, Issue 05, June, 2026

which leads to the well-known trigonometric identity:

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$.

The educational value of this task lies in demonstrating that geometric drawings can be used to derive mathematical formulas. Its developmental value is associated with the resolution of emerging difficulties through productive thinking. Moreover, it provides a means of assessing the depth of students' theoretical knowledge and their ability to apply it in practice.

Theorem 1

If two sides of a triangle are equal, then the angles opposite those sides are also equal.

Given: Triangle ABC, where AC = BC.

To prove: $\angle A = \angle B$.

Proof. Let triangle ABC be an isosceles triangle with base AB. By the first congruence criterion for triangles, the corresponding parts of congruent triangles are equal. Since AC = BC and side CC is common, the base angles opposite the equal sides are equal.

Therefore:

$\angle A = \angle B$.

This theorem demonstrates how logical reasoning and geometric visualization contribute to productive mathematical thinking.

Theorem 2

Through any point on a line, one and only one perpendicular line can be drawn.

Proof. Let line a be given, and let point A lie on line a. Consider one of the rays of line a originating at point A and denote it by a_1. Construct a 90° angle from ray a_1, producing ray b_1. The line containing ray b_1 is perpendicular to line a.

Assume that there exists another line passing through point A and perpendicular to line a. Let its corresponding ray be denoted by c_1.

Since both $\angle(a_1, b_1)$ and $\angle(a_1, c_1)$ are right angles and lie in the same half-plane, uniqueness of angle construction implies that b_1 and c_1 coincide. Therefore, no second distinct perpendicular line can exist through point A. Hence, through a given point on a line, exactly one perpendicular line can be drawn.

This theorem particularly contributes to productive thinking because students must rely on logical reasoning rather than computational procedures. The key idea requiring productive thinking is the proof of the uniqueness of the perpendicular line.

Conclusion

The development of mathematical proof skills requires a high level of productive thinking. Several factors should be taken into account:

- the level of students' thinking skills;
- their ability to identify essential features of mathematical concepts;
- the quantity and quality of acquired knowledge;
- familiarity with different methods of proof;
- the ability to express proofs both orally and in written form.

Teaching proof techniques is essential because the ability to construct logically sound arguments is one of the fundamental components of mathematical competence. The renowned Russian psychologist Vladimir A. Krutetskii thoroughly discussed the psychological aspects of developing mathematical abilities in his work *Psychology of Mathematical Abilities in Schoolchildren*.

Many graduates continue to experience difficulties in understanding the necessity of theorem proofs and reasoning about converse statements. Therefore, special attention should be paid to developing students' problem-solving and proof-construction skills, which are integral components of productive thinking. Achieving this objective requires teachers to possess strong methodological competence and professional knowledge that meet contemporary educational standards

References

1. Resolution of the President of the Republic of Uzbekistan No. PQ-4708, dated May 7, 2020, On Measures to Improve the Quality of Education in Mathematics and Develop Scientific Research in the Field of Mathematics.
2. Alikhonov, S. (2008). Methods of Teaching Mathematics. Tashkent: O‘qituvchi Publishing House.
3. Avlayev, O. U., Jo‘rayeva, S. N., & Mirzayeva, S. P. (2017). Teaching Methods: Educational and Methodological Manual. Tashkent: Navro‘z Publishing House.
4. Xaydarov, B. Q. (2020). Mathematics, Grade 5: Textbook for the 5th Grade of General Secondary Schools (Part 1) (3rd revised and expanded edition). Tashkent.

5. Xaydarov, B. Q. (2020). Mathematics, Grade 5: Textbook for the 5th Grade of General Secondary Schools (Part 2) (3rd revised and expanded edition). Tashkent.
6. Ishmukhamedov, R., Abduqodirov, A., & Pardaev, A. (2008). Innovative Technologies in Education: A Guide for Teachers and Educators of Educational Institutions. Tashkent.
7. Ochilov, M., & Ochilova, N. (2008). Higher Education Pedagogy: A Textbook for Higher Educational Institutions. Tashkent.
8. Ro‘ziyeva, D. I. (2016). Modeling Professional Activity. Tashkent: Tashkent State Pedagogical University.
9. Tojiyev, M., Barakayev, M., & Xurramov, A. (2015). Design of Educational Lessons in the Methodology of Teaching Mathematics. Tashkent.
10. Decree of the President of the Republic of Uzbekistan On the Strategy for the Development of the Education and Upbringing System.
11. Hasanov, U. (2021). Methodology of Teaching Mathematics. Tashkent.
12. Xoliqova, M., & Qodirova, D. (2022). Pedagogical Technologies and Interactive Methods. Tashkent.
13. Karimova, D. (2023). Modern Approaches to Teaching Mathematics in the Process of Independent Learning. Innovative Education Journal.
14. www.ziyonet.uz
15. www.edu.uz
16. www.geogebra.org